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The flexural rigidity of a thin plate reinforced with periodic systems of separated rods☆

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ABSTRACT

A two-dimensional model of the flexure of a thin plate, reinforced with periodic families of separated thin rods, symmetrical about the middle plane, is constructed. Since the rods only interact through the pliable matrix material, the algorithm for constructing the asymptotics is essentially different from the classical procedure in the theory of composite plates and leads to new results. Explicit formulae are obtained for the coefficients of the fourth order differential equation which arises.

1. A composite plate

A plate of thickness $2h > 0$ is specified by the relation

$$\Omega_h = \{(y,z) : y = (y_1,y_2) \in \omega, \quad z \in (-h,h)\} \tag{1.1}$$

in the Cartesian system of coordinates $x = (y, z)$. Here, ω is the domain in a plane bounded by a simple smooth closed contour $\partial\omega$. We make the parameter h and the coordinates y_1, y_2 and z dimensionless by scaling, and we separate out the layers $\sum_h^{(k)} (k = 0, \pm 1, \pm 2, \ldots, \pm K)$ which are arranged symmetrically about the middle plane of the plate

$$\begin{aligned}
&\Sigma_h^{(0)} = \{(y,z) : y \in \omega, \quad z \in (-ha_0, ha_0)\} \\
&\Sigma_h^{(\pm k)} = \{(y,z) : y \in \omega, \quad \pm z \in (ha_{k-1}, ha_k)\}, \quad k = 1,2,\ldots,K \\
&0 \le a_0 < a_1 < \ldots < a_K = 1
\end{aligned}$$

When $a_0 > 0$, the number of layers is odd and, when $a_0 = 0$, it is even since the layer $\sum_h^{(0)}$ is missing. The plate (1.1) is pierced by periodic families Π_h^k of circular rods Π_h^{kj}, $(j = 0, \pm 1, \pm 2, \ldots \pm J)$ with axes in the middle plane of a layer $\sum_h^{(k)}$ at a distance of hs_k from one another (Fig. 1). The rods in the upper layer $\sum_h^{(K)}$ are parallel to the y_1 axis and the rods in the layer $\sum_h^{(k)}$ are at an angle α_k to this axis and, in particular, $\alpha_K = 0$. We now introduce the Cartesian systems of coordinates $x^k = (y_1^k, y_2^k, z^k)$ with the origin $x = x_0^{(k)}$ on the axis of the rod Π_h^{k0} such that the y_1^k axis is parallel to the z axis and the z^k axis is parallel to the axes of the rods Π_h^{kj}. The transition from system x to the system x^k is achieved using the orthogonal transformation

$$x \mapsto x^k = \theta^k(x - x_0^{(k)}), \quad \theta^k = \begin{Vmatrix} 0 & 0 & 1 \\ \sin\alpha_k & \cos\alpha_k & 0 \\ \cos\alpha_k & -\sin\alpha_k & 0 \end{Vmatrix} \tag{1.2}$$

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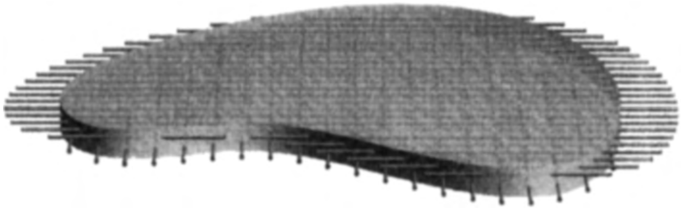

Fig. 1.

The systems of rods and the matrix-filler are defined by the formulae

$$\Pi_h^{kj} = \left\{(y^k, z^k) \in \Omega_h : |(y_1^k)^2 + (y_2^k - hjs_k)^2| < hR_k\right\}$$

$$R_0 < a_0, \quad R_k < \frac{1}{2}\min\{s_k, a_k - a_{k-1}\}; \quad \Pi_h^k = \bigcup_j \Pi_h^{kj}, \quad \Pi_h = \bigcup_{k,j} \Pi_h^{kj} \tag{1.3}$$

The isotropic material of the rods Π_h^{kj} has the Lamé constants λ_k and μ_k and the matrix material $\tilde{\Omega}_h$ has the Lamé constants $\Lambda = h\tilde{\lambda}$ and $M = h\tilde{\mu}$, where h is the same dimensionless parameter as in formula (1.1) and the quantities λ_k, μ_k, $\tilde{\mu}$ and $\tilde{\lambda}$ are comparable in their order of magnitude. The design which has been described is a mathematical model of a composite plate consisting of isotropic rigid fibres Π_h^{kj} and a matrix $\tilde{\Omega}_h$ made of a weaker but also homogeneous and isotropic elastic material. Hence, the cylindrical stiffness $O(hh^3)$ of the plate $\tilde{\Omega}_h$ is comparable in order of magnitude with the flexural stiffness $O(h^4)$ of each isolated rod Π^{kj} which, in totality, take the main load on themselves, while the matrix material serves as a filler.

Composite materials of this kind are encountered in modern engineering,[1] and the absence of direct bonds between the fibres is explained by the technological preparation process or by attempts to reduce the price of the product. In reinforced concrete structures, the reinforcement, consisting of families of rigid rods, is conventionally welded at the contact points and, in this sense, the rods are found to be joined, unlike the separated rods studied in this paper. Standard asymptotic structures (see for example, Ref. 2, Ch. 8, Ref. 3, Ch. 6, etc.) are suitable, in the case of families of connected rods, for the approximate description of the stress-strain state of a reinforced plate, and the derived and substantiated asymptotic formulae show that systems of rods that have been soldered into a united mesh absorb the main part of the load, while the role of the filler, the matrix, is of little importance and barely appears in the algorithm for the constructing of the asymptotics. If, however, the rods are separated, that is, they do not touch but are connected into a united whole solely by the filler, the standard asymptotic procedures do not work, the role of the filler increases considerably and the problem requires a new modified asymptotic analysis. We emphasize that the contact between the rods and the matrix material is assumed to be ideal and questions of fracture (peeling) are not touched upon.

The interaction of the rods exclusively through the soft matrix has several consequences. To begin with, for natural reasons a composite plate is weakly resistant to shear loads in the $z = 0$ plane, which is reflected in the absence of ellipticity of the system of equations for the plane stress state of the plate (Section 7). At the same time, in a situation of "pure" flexure of a plate considered here,with a supplementary geometric condition (Section 2), the limiting fourth order equation retains its ellipticity and the Dirichlet boundary value problem retains its unique solvability (Section 5).

The second feature of a composite plate with separated rods is the insertion of a small parameter into the model problem arising for the periodicity cell. As a result, it is necessary to modify the procedure for constructing of the asymptotic forms (Section 3) and the formulae for the coefficients of the limiting differential operator, both the general (Section 5) and the specific (Section 6), differ from the known coefficients in the case of a plate reinforced by a network of connected rods.

In the case considered

$$M \ll \mu, \quad \mu \approx \mu_k, \quad h = H/L \ll 1$$

where the thickness H and the length L are the characteristic overall dimensions of the plate and the rods, and the above mentioned cylindrical and flexural stiffnesses acquire the same order of magnitude subject to the condition that $M(H/L)^3 \approx \mu(H/L)^4$, that is, $L \approx H\mu/M$. The last relation prescribes the range of variation in the longitudinal dimensions of the plate and the lengths of the rods over which the asymptotic theory that has been developed "works". If it is turns out $M(H/L)^3 \gg \mu(H/L)^4$ and $L \gg H\mu/M$, then the standard asymptotic constructions of the theory of averaging are suitable.

2. Mathematical formulation of the problem

The displacement vector u satisfies the equilibrium equations

$$-\mu_k \Delta_x u^{k,j}(x) - (\lambda_k + \mu_k)\operatorname{grad}\operatorname{div} u^{k,j}(x) = e_{(3)} f(y), \quad x \in \Pi_h^{kj} \tag{2.1}$$

$$-h\tilde{\mu}\Delta_x \tilde{u}(x) - h(\tilde{\lambda} + \tilde{\mu})\operatorname{grad}\operatorname{div}\tilde{u}(x) = e_{(3)} f(y), \qquad x \in \tilde{\Omega}_h \tag{2.2}$$

Here, $\tilde{\mu}$ and $u^{k,j}$ are the contractions of the vector u in $\tilde{\Omega}_h$ and Π_h^{kj} respectively, f is the mass force and $e_{(k)}$ is the unit vector along the x^k axis. The lateral surface $\Gamma_h = \partial\omega \times (-h, h)$ of the plate Ω_h is rigidly changed:

$$u(x) = 0, \quad x \in \Gamma_h \tag{2.3}$$

The external forces

$$\sigma_{i3}(\tilde{u};y,\pm h) = 0, i = 1,2, \quad \sigma_{33}(\tilde{u};y,\pm h) = \pm hg(y), \quad x \in \partial\Omega_h \backslash \Gamma_h \tag{2.4}$$

are applied to the bottom of the plate, where σ_{ij} are the Cartesian components of the stress tensor

$$\sigma_{ij}(\tilde{u};x) = h\tilde{\lambda}\delta_{i,j}(\varepsilon_{11}(\tilde{u};x) + \varepsilon_{22}(\tilde{u};x) + \varepsilon_{33}(\tilde{u};x)) + 2h\tilde{\mu}\varepsilon_{ij}(\tilde{u};x) \tag{2.5}$$

Conditions of ideal contact

$$\tilde{u}(x) = u^{k,j}(x), \quad \sigma^{(\nu)}(\tilde{u};x) = \sigma^{(\nu)}(u^{k,j};x), \quad x \in \partial\tilde{\Omega}_h \cap \Pi_h^{kj} \tag{2.6}$$

are imposed on the sides of the rods in contact with the matrix.

We will denote a scalar product in the scalar or vector Lebesgue space $L_2(\gamma)$ by $(,)\gamma$ and the space of Sobolev functions satisfying condition (2.3) by $\overset{\circ}{H}{}^1(\Omega_h;\Gamma_h)$. The variational formulation of problem (2.1)–(2.4), (2.6), corresponding to the problem of the minimizing of the strain potential energy, has the form.[4,5]

$$\left(\sigma_{pq}(u),\varepsilon_{pq}(\upsilon)\right)_{\Omega_h} = (f,\upsilon)_{\Omega_h} + h(g,\upsilon)_{\partial\Omega_h\backslash\Gamma_h}, \quad \upsilon \in \overset{\circ}{H}{}^1(\Omega_h;\Gamma_h) \tag{2.7}$$

Henceforth, summation is carried out over the repeated indices $p, q = 1, 2, 3$. When $\upsilon = u$, the left-hand side of the integral identity (2.7) is twice the elastic energy of the plate.

We will assume that, for at least two families Π_h^{jk}, the axes of the rods are formed by crossing lines, that is, $\alpha_l \neq 0$ for some $l \neq K$. We also assume that the plate has a periodic structure, which means that it is possible to choose a general periodicity cell with dimensions $O(h)$. We will now formulate the corresponding geometrical constraints. We introduce the set κ_1 of numbers $k = 0, 1, ..., K$ and $k \neq l$ such that $\alpha_k \neq 0, \alpha_k \neq \pi$, and, also, the set κ_2 of numbers $k = 0. 1, ..., K - 1$ such that $\alpha_k \neq \pm\pi/2$. Suppose

$$S^1 = |(\sin\alpha_l)^{-1}s_l|, \quad S^2 = s_k$$
$$S_k^1 = |(\sin\alpha_k)^{-1}s_k| \ \text{при} \ k \in \varkappa_1, \quad S_k^1 = |(\cos\alpha_k)^{-1}s_k| \ \text{при} \ k \in \varkappa_2$$

We will require that all of the numbers $S_k^i/S^i (i = 1, 2, k \in {}_i)$ turn out to be simple fractions $m_k^{(i)}/n_k^{(i)}$ and determine the cell Q_h, denoting the least common multiple of the numbers $m_i^{(k)}$, k_i by P_i. We cover the plane of the plate Ω_h with a mesh of rectangles of size $b_1h \times b_2h$, where $b_i = P_iS^i$ and construct parallelepipeds of dimensions $b_1h \times b_2h \times 2h$, the family of which covers the whole plate Ω_h. Each of the parallelepipeds includes different and identically arranged parts $G_h^{(k,j)}$ of the rods Π_h^{kj}. Here, a periodicity cell Q_h can contain several fragments of the same rod from the same set Π_h^k (the periodicity cell of the system shown in Fig. 2 contains two unequal fragments of the rods from the system Π_h^2 and four congruent fragments of the equal rods of the system Π_h^1). Note that, in many cases, for a small number K of directions of reinforcement of the plate, checking that the conditions for the existence of a periodicity cell are satisfied is elementary (see Section 6).

We will now consider the pure flexure of a composite plate which is ensured by the following geometric and physical conditions. First, the families $\Pi_h^{\pm k}$ contain identical and undirectional rods, that is,

$$R_k = R_{-k}, \quad s_k = s_{-k}, \quad \alpha_k = \alpha_{-k}$$

Second, the materials of the rods Π_h^{kj} and Π_h^{-kj} are identical. In this case, the vector

$$(-u_1(y,-z), -u_2(y,-z), u_3(y,-z))$$

satisfies the same problem as the vector $u((y, z)$, and this means, in view of the uniqueness of the solution, that the following equalities hold.

$$u_p(y,z) = -u_p(y,-z), \quad p = 1,2, \quad u_3(y,z) = u_3(y,-z)$$

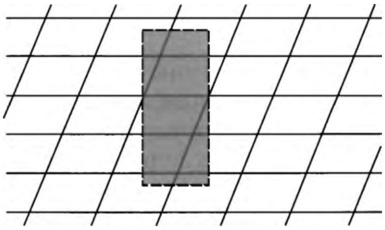

Fig. 2.

3. Leading asymptotic terms

For the displacement vector, we take the truncated asymptotic expansion of the theory of thin plates (see, for example, Ref. 3, Ch. 4)

$$h^{-2}U^{-2}(y)+h^{-1}U^{-1}(y,\zeta)=h^{-2}w(y)e_{(3)}-h^{-1}\zeta\left(\frac{\partial w}{\partial y_1}(y)e_{(1)}+\frac{\partial w}{\partial y_2}(y)e_{(2)}\right) \tag{3.1}$$

Here, $\varsigma = h^{-1}z$ is the extended transverse coordinate (the fast variable) and the function w is the mean deflection of the plate.

We rewrite formula (3.1) in the coordinates x^k attached to the rod Π_h^{kj} and refine it, adding the next asymptotic term in order of magnitude derived from the theory of rods (Ref. 3, Ch. 3). It is not required in the expansion of the solution for the plate on account of the smallness of the Lamé coefficients $\Lambda = h\tilde{\lambda}$ and $M = h\tilde{\mu}$ of the filler. We obtain

$$\begin{aligned}&h^{-2}U^{-2,k}(y_2^k,z^k)+h^{-1}U^{-1,k}(y_2^k,z^k,\eta_1^k)+h^0U^{0,k}(y_2^k,z^k,\eta^k)=h^{-2}w^k(y_2^k,z^k)e_{(1)}^k-\\&-h^{-1}(\eta_1^k-h^{-1}x_{0\,3}^k)\left(\frac{\partial w^k}{\partial y_2^k}(y_2^k,z^k)e_{(2)}^k+\frac{\partial w^k}{\partial z^k}(y_2^k,z^k)e_{(3)}^k\right)+h^0U^{0,k}(y_2^k,z^k,\eta^k)\end{aligned} \tag{3.2}$$

Here, $w^k(y_2^k, z^k) = w(y)$, $\eta^k = h^{-1}y^k$, x_0^k is a point on the axis of a rod of the family Π_h^k and $e_{(i)}^k$ and $e_{(3)}^k$ are unit vectors of the y_i^k and z^k axes.

We introduce the extended coordinates in the matrix and in the rod, connected by the relation

$$\xi^k=\theta^k\xi;\quad \xi=(\eta,\zeta)=\left(h^{-1}y,h^{-1}z\right),\quad \xi^k=(\eta^k,\zeta^k)=\left(h^{-1}y^k,h^{-1}z^k\right)$$

Substituting the sums (3.2) and the contact conditions (2.6) into Eq. (2.1), we choose the coefficients of like powers of the small parameter h. As a result, we obtain the problem for determinings the term $U^{0,k}$ which satisfies the periodicity conditions with respect to the variable η_k in the cell $Q_1 = b_1 \times b_2$. The problem admits of an explicit solution

$$U^{0,k}(y_2^k,z^k,\eta^k)=X^k(\eta^k)D_kw^k(y_2^k,z^k)$$

$$X^k(\eta^k)=\begin{Vmatrix}(\eta_2^k)^2/2 & 0 & (\eta_2^k)^2/2+V_1^k(\eta^k)\\ -\eta_1^k\eta_2^k & 0 & -\eta_1^k\eta_2^k+V_2^k(\eta^k)\\ 0 & W^k(\eta^k) & 0\end{Vmatrix},\quad D_k=\left(\frac{\partial^2}{\partial(y_2^k)^2},\sqrt{2}\frac{\partial^2}{\partial y_2^k\partial z^k},\frac{\partial^2}{\partial(z^k)^2}\right) \tag{3.3}$$

Here W^k and $V^k = (V_1^k, V_2^k)$ are the solutions of two model (antiplane and plane) problems in the circle B^k of radius R_k

$$\Delta W^k(\eta^k)=0,\quad \eta^k\in B^k,\quad \mu_k\partial_{\nu^k}W^k(\eta^k)=-\sqrt{2}\mu_k\nu_2^k\left(\eta^k\right)\eta_1^k,\quad \eta^k\in\partial B^k \tag{3.4}$$

$$\begin{aligned}&\mu_k\Delta_{\eta^k}V^k(\eta^k)+(\lambda_k+\mu_k)\operatorname{grad}\operatorname{div}V^k(\eta^k)=0,\quad \eta^k\in B^k\\&\sigma_{1i}(V^k;\eta^k)\nu_i^k(\eta^k)=0,\quad \sigma_{2i}(V^k;\eta^k)\nu_i^k(\eta^k)=-2\mu_k\nu_2^k(\eta^k)\eta_1^k,\quad \eta^k\in\partial B^k\end{aligned} \tag{3.5}$$

$\sigma_{pq}(V^k;\eta^k)$ are the stresses calculated at the point η^k in the extended coordinates along the displacement vector V^k and $\nu^k(\eta^k)$ is the outward normal to the circle ∂B^k.

4. Taking the limit in the energy functional

We will derive an integral identity in the two-dimensional model of a composite plate and define the trial function Ψ_h by a formula which imitates asymptotic representation (3.2)

$$\Psi_h(y,z)=h^{-2}\Psi^{-2}(y)+h^{-1}\Psi^{-1}(y,\zeta)+h^0\Psi^0(y,z,\xi) \tag{4.1}$$

$$\Psi^{-2}(y)=\varphi(y)e_{(3)},\quad \Psi^{-1}(y,\zeta)=\zeta\left(\frac{\partial\varphi}{\partial y_1}(y)e_{(1)}+\frac{\partial\varphi}{\partial y_2}(y)e_{(2)}\right) \tag{4.2}$$

$$\Psi^{0,k}(y_2^k,z^k,\eta^k)=X^k(\eta^k)D\left(\frac{\partial}{\partial y_2^k},\frac{\partial}{\partial z^k}\right)\varphi^k(y_2^k,z^k) \tag{4.3}$$

Here, the notation for the functions in the local system of coordinates has been taken from Section 2. Moreover, the field (4.3) is continued smoothly in an arbitrary manner from the rod into the matrix. By $\varphi \in C_c^\infty(\omega)$, we mean an infinitely differentiable function with a compact carrier in the domain ω and, moreover, $\varphi^k(y_2^k, z^k) = \varphi(y)$. We calculate the leading term of the asymptotic form of the energy integral $\sigma_{pq}(U_h)$, $\varepsilon_{pq}(\Psi_h)_{\Omega_h}$ when $h \to 0$; here

$$U_h=h^{-2}U^{-2}+h^{-1}U^{-1}+h^0\sum_{k,j}U^{0,k} \tag{4.4}$$

The summation over the indices k and j is carried out over all the rods in the composite plate.

We use the notation

$$\partial_2 w^k = \frac{\partial w^k}{\partial y_2^k}, \quad \partial_3 w^k = \frac{\partial w^k}{\partial z^k}, \quad U_{i,j}^{0,k} = \frac{\partial U_i^{0,k}}{\partial \eta_j^k}$$

According to formula 91.6) and definition (2.5) of the stress tensor, we find

$$\begin{gathered}
\sigma_{11}(U_h) = h^{-1}\sum_{k,j}\Bigl(\eta_1^k\Bigl(\lambda_k\bigl(\partial_2^2 w^k + \partial_3^2 w^k\bigr) + 2\mu_k\bigl(\sin^2\alpha_k\partial_2^2 w^k + \sin 2\alpha_k\partial_2\partial_3 w^k + \cos^2\alpha_k\partial_3^2 w^k\bigr)\Bigr) + \\
+ \lambda_k\bigl(U_{1,1}^{0,k} + U_{2,2}^{0,k}\bigr) + \mu_k\bigl(2\sin^2\alpha_k U_{2,2}^{0,k} + \sin 2\alpha_k U_{3,2}^{0,k}\bigr)\Bigr) + \ldots \\
\sigma_{22}(U_h) = h^{-1}\sum_{k,j}\Bigl(\eta_1^k\Bigl(\lambda_k(\partial_2^2 w^k + \partial_3^2 w^k\bigr) + 2\mu_k\bigl(\cos^2\alpha_k\partial_2^2 w^k - \sin 2\alpha_k\partial_2\partial_3 w^k + \sin^2\alpha_k\partial_3^2 w^k\bigr)\Bigr) + \\
+ \lambda_k\bigl(U_{1,1}^{0,k} + U_{2,2}^{0,k}\bigr) + \mu_k\bigl(2\cos^2\alpha_k U_{2,2}^{0,k} - \sin 2\alpha_k U_{3,2}^{0,k}\bigr)\Bigr) + \ldots \\
\sigma_{12}(U_h) = \sigma_{12}(U_h) = h^{-1}\sum_{k,j}\Bigl(\mu_k\eta_1^k\bigl(\sin 2\alpha_k\partial_2^2 w^k + 2\cos 2\alpha_k\partial_2\partial_3 w^k - \sin 2\alpha_k\bigr)U_{3,2}^{0,k} + \\
+ \mu_k\bigl(\sin 2\alpha_k U_{2,2}^{0,k} + \cos 2\alpha_k U_{3,2}^{0,k}\bigr)\Bigr) + \ldots \\
\sigma_{33}(U_h) = h^{-1}\sum_{k,j}\Bigl((\lambda_k + 2\mu_k)U_{1,1}^{0,k} + \lambda_k U_{2,2}^{0,k}\Bigr) + \ldots \\
\sigma_{13}(U_h) = \sigma_{31}(U_h) = h^{-1}\sum_{k,j}\mu_k\Bigl(\sin\alpha_k\bigl(U_{1,2}^{0,k} + U_{2,1}^{0,k}\bigr) + \cos\alpha_k U_{3,1}^{0,k}\Bigr) + \ldots \\
\sigma_{23}(U_h) = \sigma_{32}(U_h) = h^{-1}\sum_{k,j}\mu_k\Bigl(\cos\alpha_k\bigl(U_{1,2}^{0,k} + U_{2,1}^{0,k}\bigr) - \sin\alpha_k U_{3,1}^{0,k}\Bigr) + \ldots
\end{gathered} \tag{4.5}$$

Henceforth, dots replace terms of higher orders of smallness.

The components of the stress tensor are only of an order of magnitude h^{-1} in the case of the rods Π_h^{kj} and, for the matrix, they are uniformly bounded. Hence, only the deformations of the rods are subsequently necessary

$$\begin{gathered}
\varepsilon_{11}(\Psi_h) = h^{-1}\sum_{k,j}\Bigl(\eta_1^k\bigl(\sin^2\alpha_k\partial_2^2\varphi^k + \sin 2\alpha_k\partial_2\partial_3\varphi^k + \cos^2\alpha_k\partial_3^2\varphi^k\bigr) + \\
+ \sin^2\alpha_k\Psi_{2,2}^{0,k} + \sin\alpha_k\cos\alpha_k\Psi_{3,2}^{0,k}\Bigr) + \ldots \\
\varepsilon_{22}(\Psi_h) = h^{-1}\sum_{k,j}(\eta_1^k\bigl(\cos^2\alpha_k\partial_2^2\varphi^k - \sin 2\alpha_k\partial_2\partial_3\varphi^k + \sin^2\alpha_k\partial_3^2\varphi^k\bigr) + \\
+ \cos^2\alpha_k\Psi_{2,2}^{0,k} - \sin\alpha_k\cos\alpha_k\Psi_{3,2}^{0,k}) + \ldots \\
\varepsilon_{12}(\Psi_h) = \varepsilon_{12}(\Psi_h) = h^{-1}\sum_{k,j}\Bigl(\eta_1^k\bigl(\sin 2\alpha_k\partial_2^2\varphi^k + 2\cos 2\alpha_k\partial_2\partial_3\varphi^k - \\
- \sin 2\alpha_k\Psi_{3,2}^{0,k}\bigr) + \mu_k\bigl(\sin 2\alpha_k\Psi_{2,2}^{0,k} + \cos 2\alpha_k\Psi_{3,2}^{0,k}\bigr)\Bigr) + \ldots \\
\varepsilon_{33}(\Psi_h) = h^{-1}\sum_{k,j}\Psi_{1,1}^{0,k} + \ldots \\
\varepsilon_{13}(\Psi_h) = \varepsilon_{31}(\Psi_h) = h^{-1}\sum_{k,j}\Bigl(\sin\alpha_k\bigl(\Psi_{1,2}^{0,k} + \Psi_{2,1}^{0,k}\bigr) + \cos\alpha_k\Psi_{3,1}^{0,k}\Bigr) + \ldots \\
\varepsilon_{23}(\Psi_h) = \varepsilon_{32}(\Psi_h) = h^{-1}\sum_{k,j}\Bigl(\cos\alpha_k\bigl(\Psi_{1,2}^{0,k} + \Psi_{2,1}^{0,k}\bigr) - \sin\alpha_k\Psi_{3,1}^{0,k}\Bigr) + \ldots \\
\Psi_{i,j}^{0,k}(y_2^k, z^k, \eta^k) = \frac{\partial\Psi_i^{0,k}}{\partial\eta_j^k}(y_2^k, z^k, \eta^k)
\end{gathered} \tag{4.6}$$

Using expressions (4.5) and (4.6), we obtain

$$\begin{aligned}&\big(\sigma_{ij}(u),\varepsilon_{ij}(\upsilon)\big)_{\Omega_h}-(f,\upsilon)_{\Omega_h}-h(g,\upsilon)_{\partial\Omega_h\setminus\Gamma_h}=\\&=\sum_{Q_h\in\{Q_h\}}\Big[\big(\sigma_{ij}(u),\varepsilon_{ij}(\upsilon)\big)_{\Omega_h\cap Q_h}-(f,\upsilon)_{\Omega_h\cap Q_h}-h(g,\upsilon)_{(\partial\Omega_h\setminus\Gamma_h)\cap\overline{Q_h}}\Big]=\\&=h^{-1}\sum_k\Big\{E^k(w,\varphi;\omega)-b_1b_2(f+g,\varphi)_\omega\Big\}+O(h^0)\end{aligned}\tag{4.7}$$

The summation is carried out over all systems of rods and

$$\begin{aligned}E^k(w,\varphi;\omega):=&\int\limits_{|\eta^k|<R_k}((\eta_1^k)^2\Big(\lambda_k(\partial_2^2w+\partial_3^2w)(\partial_2^2\varphi^k+\partial_3^2\varphi^k)+\\&+2\mu_k(\partial_2^2w\partial_2^2\varphi^k+2\partial_2\partial_3w\partial_2\partial_3\varphi^k+\partial_3^2w\partial_3^2\varphi^k)\Big)+\eta_1^k\Big(\lambda_k(\partial_2^2w+\partial_3^2w)(\Psi_{1,1}^{0,k}+\Psi_{2,2}^{0,k})+\\&+2\mu_k(\partial_2^2w\Psi_{2,2}^{0,k}+\partial_2\partial_3w\Psi_{3,2}^{0,k})+\lambda_k(U_{1,1}^{0,k}+U_{2,2}^{0,k})(\partial_2^2\varphi^k+\partial_3^2\varphi^k)\Big)+\\&+\lambda_k(U_{1,1}^{0,k}+U_{2,2}^{0,k})(\Psi_{1,1}^{0,k}+\Psi_{2,2}^{0,k})+\mu_k\Big(2U_{1,1}^{0,k}\Psi_{1,1}^{0,k}+2U_{2,2}^{0,k}\Psi_{2,2}^{0,k}+\\&+(U_{2,1}^{0,k}+U_{1,2}^{0,k})(\Psi_{2,1}^{0,k}+\Psi_{1,2}^{0,k})+U_{3,1}^{0,k}\Psi_{3,1}^{0,k}+U_{3,2}^{0,k}\Psi_{3,2}^{0,k}\Big)dy\end{aligned}\tag{4.8}$$

We now write the quantity (4.8) in the form

$$E^k(w,\varphi;\omega)=\Big(S(U^{0,k},w),T(\Psi^{0,k},\varphi)\Big)_\omega$$

$$S(U^{0,k},w)=(S_1,S_2,S_3,S_4,S_5,S_6)$$

$$S_1=(\lambda_k+2\mu_k)U_{1,1}^{0,k}+\lambda_k(U_{2,2}^{0,k}+\eta_1\partial_2^2w)+\lambda_k\eta_1\partial_3^2w$$

$$S_2=\lambda_kU_{1,1}^{0,k}+(\lambda_k+2\mu_k)(U_{2,2}^{0,k}+\eta_1\partial_2^2w)+\lambda_kU_{1,1}^{0,k}$$

$$S_3=\sqrt{2}\mu_k(U_{1,2}^{0,k}+U_{2,1}^{0,k}),\quad S_4=\sqrt{2}\mu_kU_{3,1}^{0,k},\quad S_5=2\sqrt{2}\mu_k(U_{1,2}^{0,k}+\eta_1\partial_2\partial_3w)$$

$$S_6=\lambda_kU_{1,1}^{0,k}+\lambda_k(U_{2,2}^{0,k}+\eta_1\partial_2^2w)+(\lambda_k+2\mu_k)U_{1,1}^{0,k}$$

$$T(\Psi^{0,k},\varphi)=\left(\Psi_{1,1}^{0,k},\Psi_{2,2}^{0,k}+\eta_1\partial_2^2w,\frac{1}{\sqrt{2}}(\Psi_{1,2}^{0,k}+\Psi_{2,1}^{0,k}),\frac{1}{\sqrt{2}}\Psi_{3,1}^{0,k},\frac{1}{\sqrt{2}}(\Psi_{1,2}^{0,k}+\eta_1\partial_2\partial_3w),\eta_1\partial_3^2w\right)$$

Estimation of the residue in the asymptotic formula (4.7) is ensured by the Korn inequality for a composite plate with a strictly periodic structure, derived using the standard scheme.[8]

5. The limiting problem of the bending of a plate

The integral identity for the deflection w has the form

$$\frac{1}{b_1b_2}\sum_k E^k(w,\varphi;\omega)=(F,\varphi)_\omega,\quad \varphi\in\overset{\circ}{H}{}^2(\omega);\quad F=2(f+g)\tag{5.1}$$

$\overset{\circ}{H}{}^2(\omega)$ is the space of Sobolev functions w satisfying the conditions

$$w(y)=0,\quad \nabla w(y)=0,\quad x\in\Gamma_h\tag{5.2}$$

We now calculate the components of the left-hand side of the equality (5.1). By to formulae (4.3) and (3.3), we have

$$E^k(w,\varphi;\omega)=\Big(M^kD_kw,D_k\varphi\Big)_\omega\tag{5.3}$$

Here

$$M^k=\operatorname{diag}\Big\{0,M_2^k,M_3^k\Big\}$$

$$M_2^k=\int\limits_{B^k}\left(\mu^k\left(\frac{\partial W^k}{\partial\eta_1^k}\right)^2+\mu^k\left(\frac{\partial W^k}{\partial\eta_2^k}+\sqrt{2}\eta_1\right)^2\right)d\eta$$

$$M_3^k = \int\limits_{B^k}\left(\lambda^k\left(\frac{\partial V_1^k}{\partial\eta_1^k}+\frac{\partial V_2^k}{\partial\eta_2^k}\right)^2+\mu^k\left(\frac{\partial V_1^k}{\partial\eta_2^k}+\frac{\partial V_2^k}{\partial\eta_1^k}\right)^2+2\mu^k\left(\frac{\partial V_1^k}{\partial\eta_1^k}\right)^2+\right.$$
$$\left.+\mu^k\left(\frac{\partial V_2^k}{\partial\eta_2^k}\right)^2+\mu^k\left(\frac{\partial V_2^k}{\partial\eta_2^k}-2\eta_1\right)^2\right)d\eta^k$$

and, for brevity, the argument η^k of the derivatives of the functions W^k, V_1^k and V_2^k is not shown.

The associated harmonic function U_0^k for the solution W^k of problem (3.4) satisfies the Cauchy–Riemann equations and, correspondingly, can be represented as follows:

$$\frac{\partial W^k}{\partial\eta_1^k}=\frac{\partial U_0^k}{\partial\eta_2^k},\quad \frac{\partial W^k}{\partial\eta_2^k}=-\frac{\partial U_0^k}{\partial\eta_1^k};\quad U_0^k(\eta^k)=C_k+\frac{1}{\sqrt{2}}(\eta_1^k)^2+\frac{1}{\sqrt{2}}U^k(\eta^k) \tag{5.4}$$

By virtue of the Cauchy–Riemann equations (5.4) and the boundary condition in problem (3.4), the tangential derivative $\partial U^k/\partial s$ is equal to zero. Consequently, the constant C_k can be chosen so that

$$\Delta_{\eta^k}U^k(\eta^k)=2,\quad \eta^k\in B^k,\quad U^k(\eta^k)=0,\quad \eta^k\in\partial B^k$$

The torsional stiffness of the circular section B^k

$$G_k = \int\limits_{B^k}\left(\left(\frac{\partial U^k}{\partial\eta_1^k}(\eta^k)\right)^2+\left(\frac{\partial U^k}{\partial\eta_2^k}(\eta^k)+\sqrt{2}\eta_1^k\right)^2\right)d\eta^k$$

is equal to $\pi R_k^4/2$.[6] Hence,

$$M_2^k = \mu^k G_k/2 = \mu^k\pi R_k^4/4 \tag{5.5}$$

The solution V^k of problem (3.4) satisfies the relations

$$\sigma_{11}(V^k;\eta^k)=\sigma_{21}(V^k;\eta^k)=\sigma_{12}(V^k;\eta^k)=0,\quad \sigma_{22}(V^k;\eta^k)=-2\mu\eta_1^k$$

Hence, by Hooke's law, we have

$$\varepsilon_{22}(V^k;\eta^k)=-\Lambda^k\eta_1^k;\quad \Lambda^k=\frac{\lambda^k+2\mu^k}{2(\lambda^k+\mu^k)}$$

$$M_3^k = \int\limits_{B^k}(\sigma_{ij}(V^k;\eta^k)\varepsilon_{ij}(V^k;\eta^k)-4\mu^k\eta_1^k\varepsilon_{22}(V^k;\eta^k)+4\mu^k(\eta_1^k)^2)d\eta^k =$$
$$=2\mu^k\int\limits_{B^k}(3\Lambda^k+2)(\eta_1^k)^2d\eta^k=\frac{\mu^k(7\lambda^k+10\mu^k)}{4(\lambda^k+\mu^k)}\pi R_k^4 \tag{5.6}$$

Formulae (5.1), (5.3), (5.5) and (5.6) imply the equation

$$L(\nabla_y)w = b_1^{-1}b_2^{-1}\sum_k L_k\left(\frac{\partial}{\partial y_2^k},\frac{\partial}{\partial z^k}\right)w :=$$
$$:=\frac{\pi}{4}b_1^{-1}b_2^{-1}\sum_k\mu^k R_k^4\left(\frac{\partial^4 w}{\partial(y_2^k)^2\partial(z^k)^2}(y)+\frac{7\lambda^k+10\mu^k}{\lambda^k+\mu^k}\frac{\partial^4 w}{\partial(z^k)^4}(y)\right)=F(y) \tag{5.7}$$

Each of the terms on the right-hand side of equality (5.3), forming the energy quadratic form on the left-hand side of the integral identity (5.1), is positive but it is not a positive definite form, since the diagonal matrix M^k has a null element on the principal diagonal. Hence, each of the operators L_k constituting the operator L in Eq. (5.7) is found to be formally positive but not elliptic. This is the decisive difference between plates reinforced with connected families of rods and plates reinforced with separated families of rods. In the case of reinforcement with a network of connected rods (the welded reinforcement in reinforced concrete, for example), the averaging procedure (Ref. 2, Ch. 8 and Ref. 3, Ch 6) at once gives an elliptic operator which enables us to find the unique solution of the problem. In the case of reinforcement with families of separated rods, each family Π_h^{jk} generates an operator L_k. It is unclear, however, whether the operator L is elliptic. The condition of crossing lines which has been introduced ensures ellipticity but, if the axes of all the rods are parallel and the condition is violated, then, as previously, the operator $L=\sum L_k$ is devoid of the derivative $\partial^4/\partial y_2^4$. It is not elliptic and the averaged problem will not be uniquely solvable. This fact reflects a simple observation: in the case of parallel rods, the bending moment is mainly absorbed by the filler which has a relatively small Young's modulus.

We will now verify the ellipticity of the operator with the additional conditions that the axes of the rods cross. The matrix $M'^k = diag\left\{M_2^k, M_3^k\right\}$ is positive and, consequently, the relation

$$D(t_1^k,t_2^k)^\top M'^k D(t_1^k,t_2^k) \ge c_k|t^k|^4, \quad c_k > 0 \tag{5.8}$$

holds for any column (τ_1^j, τ_2^j) (τ is the transposition sign). We put $M = M'^1 + M'^2 + \ldots$ and show that the equality $D^\tau MD = 0$ is only possible when $\tau = (\tau_1, \tau_2) = 0$. We select the index l for which $\alpha_1 \neq 0$ and represent the column τ in the form $\tau = c_1 e^{(1)} + c_0 e^{(l)}$, where c_0 and c_1 are constants and $e^{(1)} = (1, 0)$ and $e^{(l)} = (\cos\alpha_l, \sin\alpha_l)$ are unit vectors of the axes of the rods. According to inequality (5.8), the relation

$$0 = D(\tau)^\top MD(\tau) \ge c\left(|c_1|^4|e^{(1)}|^4 + |c_0|^4|e^{(l)}|^4\right) = c(|c_1|^4 + |c_0|^4)$$

is satisfied. As a result, we have $c_1 = c_0 = 0$ and $\tau = 0$, as was required.

So, the quadratic form $E(w,\varphi;\omega)$ is positive definite and this means that it can be designated by a scalar product in the Sobolev space $\overset{\circ}{H}{}^2(\omega)$. Riesz's theorem on the representation of a linear functional in Hilbert space establishes the unique solvability of variational problem (5.1) and the available results[7] enable one to convince oneself of the additional smoothness of the solution.

Proposition. For any left-hand side $F \in L_2(\omega)$, problem (5.1) has a unique solution $w \in H^4(\omega) \cap \overset{\circ}{H}{}^2(\omega)$ and the estimate

$$\left\|w;\, H^4(\omega) \cap \overset{\circ}{H}{}^2(\omega)\right\| \le \left\|F;\, L_2(\omega)\right\|$$

is correct.

6. Examples

Suppose there are two pairs of symmetrically arranged mutually perpendicular systems of rods (Fig. 3, the periodicity cell is shown shaded) with identical properties, that is, $\lambda^k = \lambda$, $\mu^k = \mu$, $R^k = R$ The resulting equation (5.7) then takes the form

$$s_1^{-1}s_2^{-1}\frac{\pi}{4}\mu R^4\left(\frac{7\lambda + 10\mu}{\lambda + \mu}\left(\frac{\partial^4 w}{\partial y_1^4} + \frac{\partial^4 w}{\partial y_2^4}\right) + 2\frac{\partial^4 w}{\partial y_1^2 \partial y_2^2}\right) = F$$

where s_i is a step along the rectangular mesh in the direction of the y_i axis. The same operator arises in the case when there are three families of mutually perpendicular system of rods and the radius of the central rods Π_h^{0j} is equal to $2R$.

We will now assume that a plate is reinforced with three pairs of families of rods with identical properties at an angle of 60° to one another (the scheme for the reinforcement of the plate is shown in Fig. 4) and that the distance between the neighbouring unidirectional rods is equal to s. Suppose the rods of the system Π_h^3 are directed along the y_1 axis. Searching for the dimensions of the periodicity cell in accordance with the algorithm mentioned in Section 2, we obtain

$$b_1 = 2\sqrt{3}s/3, \quad b_2 = 2s$$

We now calculate the component of the differential operator in formula (5.7), taking account of relation (1.6) and the equality

$$\theta_1 = \theta_{-1} = -\pi/6, \quad \theta_2 = \theta_{-2} = \pi/6, \quad \theta_3 = \theta_{-3} = 0$$

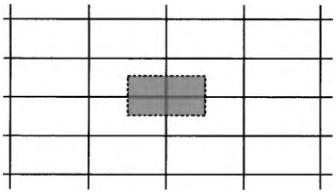

Fig. 3.

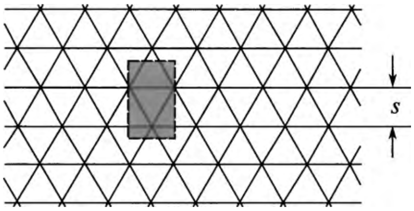

Fig. 4.

We have

$$\frac{\partial^4}{\partial(y_2^l)^2\partial(z^l)^2}=\frac{3}{16}\frac{\partial^4}{\partial y_1^4}+(-1)^{l+1}\frac{\sqrt{3}}{4}\frac{\partial^4}{\partial y_1^3\partial y_2}-\frac{1}{8}\frac{\partial^4}{\partial y_1^2\partial y_2^2}-(-1)^l\frac{\sqrt{3}}{4}\frac{\partial^4}{\partial y_1\partial y_2^3}+\frac{3}{16}\frac{\partial^4}{\partial y_2^4},\quad l=1,2$$

$$\frac{\partial^4}{\partial(z^{2l-1})^4}=\frac{1}{16}\frac{\partial^4}{\partial y_1^4}+(-1)^{l+1}\frac{3\sqrt{3}}{4}\frac{\partial^4}{\partial y_1^3\partial y_2}+\frac{9}{8}\frac{\partial^4}{\partial y_1^2\partial y_2^2}+(-1)^l\frac{\sqrt{3}}{4}\frac{\partial^4}{\partial y_1\partial y_2^3}+\frac{9}{16}\frac{\partial^4}{\partial y_2^4},\quad l=1,2$$

$$\frac{\partial^4}{\partial(y_2^3)^2\partial(z^3)^2}=\frac{\partial^4}{\partial y_1^2\partial y_2^2},\quad \frac{\partial^4}{\partial(z^3)^4}=\frac{\partial^4}{\partial y_1^4}$$

Consequently, the resulting operator (5.7) is determined from the formula

$$L=\frac{3\sqrt{3}\pi R^4}{128s^2}\frac{\mu(22\lambda+31\mu)}{\lambda+\mu}\Delta^2 w \tag{6.1}$$

The cylindrical stiffness of the homogeneous plate

$$D=\frac{h^4\tilde{\mu}(\tilde{\lambda}+\tilde{\mu})}{3(\tilde{\lambda}+2\tilde{\mu})}$$

differs in form from the factor in front of the biharmonic operator Δ^2 in formula (6.1).

In order to describe the bending of a plate reinforced with two perpendicular families of separated rods, some researchers use an equation with the operator

$$L(\nabla_y)=A\frac{\partial^4}{\partial y_1^4}+B\frac{\partial^4}{\partial y_2^4} \tag{6.2}$$

The method of "derivation" of the averaged equation serves as the reason for the absence of the mixed derivative $\partial^4/\partial y_1^2\partial y_2^2$ in the operator (6.2): the ordinary fourth order differential operators describing the bending of the individual rods from the two families are added together. The asymptotic analysis presented in the preceding sections shows that, in the resulting operator

$$L(\nabla_y)=A\frac{\partial^4}{\partial y_1^4}+2C\frac{\partial^4}{\partial y_1^2\partial y_2^2}+B\frac{\partial^4}{\partial y_2^4}$$

the coefficient C cannot degenerate into zero, that is, the operator (6.2) cannot appear.

7. The Anisotropic Korn inequality and proof of the asymptotic form

The anisotropic weighted Korn inequality (see for example, Ref. 8) is required to prove the asymptotic formulae which have been obtained. If the formulation of the problem in Section 1 is changed and it is assumed that the rods are connected and they form a periodic lattice, then the inequality.[9]

$$\begin{aligned}&\int\limits_{\tilde{\Omega}_h}\left(\sum_{j=1}^{2}\left(h\left(|u_{j,1}|^2+|u_{j,2}|^2\right)+\min\left\{h,\frac{h^2}{\rho_h^2}\right\}\left(|u_{j,3}|^2+|u_{3,j}|^2\right)+\min\left\{\frac{1}{h},\frac{1}{\rho_h^2}\right\}|u_j|^2\right)+\right.\\&\left.+h|u_{3,3}|^2+\min\left\{\frac{1}{h},\frac{h^2}{\rho_h^4}\right\}|u_3|^2\right)dydz+\\&+\sum_{k,j}\int\limits_{\Pi_h^{kj}}\left(\sum_{i=1}^{2}\left(|u_{i,1}|^2+|u_{i,2}|^2+\frac{h^2}{\rho_h^2}\left(|u_{i,3}|^2+|u_{3,i}|^2\right)+\frac{1}{\rho_h^2}|u_i|^2\right)+|u_{3,3}|^2+\frac{h^2}{\rho_h^4}|u_3|^2\right)dydz\le\\&\le C\left(E_h(u;\tilde{\Omega}_h)+h\sum_{j,k}E_h(u;\Pi_h^{kj})\right)\\&u_{k,j}=\frac{\partial u_k}{\partial y_j},\quad k=1,2,3;\quad j=1,2;\quad u_{j,3}=\frac{\partial u_j}{\partial z},\quad k=1,2,3\end{aligned} \tag{7.1}$$

holds. Here,

$$E_h(u;\Xi)=\int\limits_{\Xi}\sum_{i,j}\varepsilon_{ij}\varepsilon_{ij}dx,\quad \rho_h(y)=\min\{1,h+\operatorname{dist}(y,\partial\omega)\}$$

$\tilde{\Omega}_h$ is the matrix-filler (1.3) and the weighting factor $\rho_h(y)$ is of the order of h in the neighbourhood of the surface Γ_h of the plate Ω_h and of the order of $h^0=1$ far from Γ_h. The quantity on the left-hand side of inequality (7.1) is similar to the elastic energy of a composite plate and the constant C is independent of both the displacement field u and the parameter $h \in (0,1)$. The Korn inequality (7.1) is asymptotically exact in the sense that it is impossible to increase the order of the factor on one of the terms on its left-hand side with respect to the parameter h^{-1}. In the case of separated rods (we have now returned to the initial formulation of the problem) the unit factors on $|u_{1,2}|^2$, $|u_{2,1}|^2$ and ρ_h^{-2} when $\rho_h^{-2}|u_j|^2$ decrease to h and $h\rho_h^{-2}$ respectively (see Ref. 8). However, while this is unimportant at first glance, the change turns out to be decisive: the vector $u(x)=(y_2,-y_{1,0})$ of rotation about the z axis imparts the order of h to the integral on the left-hand side of the Korn inequality in the first case and the order of h^2 in the second case. General methods[10] enable us to derive from this fact that there is no ellipticity in the case of the averaged system of equations of the theory of elasticity describing the plane stressed state of the reinforced plate considered.

A relaxation of the weighted Korn inequality does not affect the bending component u_3: in the case when $\alpha_k \neq 0$ even if, for one of the rods, the factors on $|u_{1,2}|^2$,$|u_{2,1}|^2$ are the same as in the Korn inequality (7.1). The limiting operator L in the pure bending problem is therefore elliptic and problem (5.7), (5.2) is uniquely solvable.

In the case when $\alpha_1 = \ldots \alpha_K = 0$, all the factors on $|u_{1,2}|^2$ and $|u_{2,1}|^2$ become equal to h, and the operator (5.7) loses its ellipticity:

$$L(\nabla_y) = A\frac{\partial^4}{\partial y_1^4} + B\frac{\partial^2}{\partial y_1^2}\frac{\partial^2}{\partial y_2^2}$$

This is explained by the fact that a composite plate is weakly resistant to loads with a non-zero moment with respect to the y_1 axis. If the degree of contrast in the elastic properties of a composite plate is not taken care of and averaging is carried out using a classical scheme, then elliptic operators arise as a result, but their coefficients turn out to depend on a small parameter and the operators will be degenerate[11] when $h \to +0$ and, correspondingly, the relaxed differential properties of the solutions[12] do not enable us to prove the incorrectly performed asymptotic analysis.

An estimate of the closeness of the true and approximate solution (4.4) of the three-dimensional problem in the theory of elasticity is obtained using the standard scheme (see, for example, Ref. 3, Ch. 6) on the basis of the asymptotically exact Korn inequality for a plate reinforced with separated rods which is derived using previously developed techniques.[8]

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